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# THREE-STEP ITERATIVE SEQUENCES WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

J. K. KIM, K. H. KIM AND K. S. KIM

**ABSTRACT.** In this paper, we will give some necessary and sufficient conditions for three-step iterative sequences with errors to converge to a fixed point for asymptotically quasi-nonexpansive mappings in convex metric spaces. The results of this paper are generalizations and improvements of the corresponding results of Chang, Kim *et al.*, Liu and Xu-Noor.

## 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that  $E$  is a metric space,  $F(T)$  and  $D(T)$  are the set of all fixed points and domain of  $T$  respectively and  $\mathbb{N}$  is the set of all positive integers.

**Definition 1.1.** Let  $T : D(T) \subset E \rightarrow E$  be a mapping.

- (1) The mapping  $T$  is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in D(T).$$

- (2) The mapping  $T$  is said to be *quasi-nonexpansive* if

$$d(Tx, p) \leq d(x, p), \quad \forall x \in D(T), \forall p \in F(T).$$

- (3) The mapping  $T$  is said to be *asymptotically nonexpansive* if there exists a sequence  $k_n \in [0, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 0$  such that

$$d(T^n x, T^n y) \leq (1 + k_n)d(x, y), \quad \forall x, y \in D(T), \forall n \in \mathbb{N}.$$

- (4) The mapping  $T$  is said to be *asymptotically quasi-nonexpansive* if there exists a sequence  $k_n \in [0, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 0$  such that

$$d(T^n x, p) \leq (1 + k_n)d(x, p), \quad \forall x \in D(T), \forall p \in F(T), \forall n \in \mathbb{N}.$$

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**Remark 1.1.** From the Definition 1.1, it follows that if  $F(T)$  is nonempty, then a non-expansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

The iterative approximation problems of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings in Hilbert spaces or Banach spaces have been studied extensively by many authors. In 1973, Petryshyn-Williamson [11] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings and later, the result of [11] was extended by Ghosh-Debnath [5] to Ishikawa iterative sequences. And recently, Chang [1-3] has proved some another kinds of necessary and sufficient conditions for Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically nonexpansive mappings and Xu-Noor [14] has established a convergence theorem of three-step iterative sequences with errors for asymptotically nonexpansive mappings in uniformly convex Banach spaces. In particular, Liu [7] obtained a necessary and sufficient condition for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings in Banach spaces to converge to a fixed point and he [8] has also extended his result [7] to Ishikawa iterative sequences with errors. Furthermore, Kim *et al.* [6] extended to modified three-step iterative sequences with mixed errors.

Rhoades [12] and Naimpally-Singh [10] suggest the following open question.

**Open Question.** Can the Ishikawa iterative procedure be extended to nonlinear quasi-contractive mapping in a metric space?

Recently, Chang-Kim [4] proved convergence theorems of the Ishikawa type iterative sequences with errors for generalized quasi-contractive mappings in convex metric spaces.

The purpose of this paper is to study some necessary and sufficient conditions for three-step iterative sequences with errors to converge to fixed points for asymptotically quasi-nonexpansive mappings in convex metric spaces. The results of this paper are generalizations and improvements of the corresponding results in Chang [1-3], Ghosh-Debnath [5], Kim *et al.* [6], Liu [7-9] and Xu-Noor [14].

For the sake of convenience, we first recall some definitions and notations.

**Definition 1.2.** Let  $(E, d)$  be a metric space and  $I = [0, 1]$ . A mapping  $W : E^3 \times I^3 \rightarrow E$  is said to be a *convex structure* on  $E$  if it satisfies the following conditions : for all  $u, x, y, z \in E$  and for all  $\alpha, \beta, \gamma \in I$  with  $\alpha + \beta + \gamma = 1$ ,

- (1)  $W(x, y, z; \alpha, 0, 0) = x$ ,
- (2)  $d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z)$ .

If  $(E, d)$  is a metric space with a convex structure  $W$ , then  $(E, d)$  is called a *convex metric space* and denotes it by  $(E, d, W)$ .

**Remark 1.2.** Every linear normed space is a convex metric space, where a convex structure  $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$ , for all  $x, y, z \in E$  and  $\alpha, \beta, \gamma \in I$  with  $\alpha + \beta + \gamma = 1$ . But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [13]).

**Definition 1.3.** (1) Let  $(E, d, W)$  be a convex metric space,  $T : E \rightarrow E$  be a mapping and let  $x_1 \in E$  be a given point. Then the sequence  $\{x_n\}$  defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n y_n, u_n; a_n, b_n, c_n), \\ y_n = W(x_n, T^n z_n, v_n; \bar{a}_n, \bar{b}_n, \bar{c}_n), \\ z_n = W(x_n, T^n x_n, w_n; \hat{a}_n, \hat{b}_n, \hat{c}_n), \quad \forall n \in \mathbb{N}, \end{cases} \quad (1.1)$$

is called *the three-step iterative sequence with errors* for the mapping  $T$ , where  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{\bar{a}_n\}$ ,  $\{\bar{b}_n\}$ ,  $\{\bar{c}_n\}$ ,  $\{\hat{a}_n\}$ ,  $\{\hat{b}_n\}$  and  $\{\hat{c}_n\}$  are nine sequences in  $[0, 1]$  satisfying the following conditions:

$$a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = \hat{a}_n + \hat{b}_n + \hat{c}_n = 1, \quad \forall n \in \mathbb{N},$$

and  $\{u_n\}$ ,  $\{v_n\}$ ,  $\{w_n\}$  are three bounded sequences in  $E$ .

(2) In (1.1), if  $\hat{b}_n \equiv 0$  and  $\hat{c}_n \equiv 0$ , for all  $n = 1, 2, \dots$ , then  $z_n = x_n$ . Then the sequence  $\{x_n\}$  defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n y_n, u_n; a_n, b_n, c_n), \\ y_n = W(x_n, T^n x_n, v_n; \bar{a}_n, \bar{b}_n, \bar{c}_n), \quad \forall n \in \mathbb{N}, \end{cases} \quad (1.2)$$

is called *the Ishikawa type (or two-step) iterative sequence with errors* for the mapping  $T$ , where  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{\bar{a}_n\}$ ,  $\{\bar{b}_n\}$  and  $\{\bar{c}_n\}$  are six sequences in  $[0, 1]$  satisfying the following conditions :

$$a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1, \quad \forall n \in \mathbb{N},$$

and  $\{u_n\}$ ,  $\{v_n\}$  are two bounded sequences in  $E$ .

## 2. MAIN RESULTS

In order to obtain the main theorems, we will first prove the following lemma.

**Lemma 2.1.** Let  $(E, d, W)$  be a convex metric space,  $T : E \rightarrow E$  be an asymptotically quasi-nonexpansive mapping satisfying  $\sum_{n=1}^{\infty} k_n < \infty$  where  $\{k_n\}$  is the sequence appeared in Definition 1.1, and  $F(T)$  be a nonempty set. For a given  $x_1 \in E$ , let  $\{x_n\}$  be the three-step iterative sequence with errors defined by (1.1). Then

- (a)  $d(x_{n+1}, p) \leq (1 + k_n)^3 d(x_n, p) + h_n$ ,  $\forall p \in F(T)$ ,  $n \in \mathbb{N}$ ,  
where  $h_n = b_n(1 + k_n)\theta_n + c_n d(u_n, p)$ ,  $\theta_n = \bar{b}_n \hat{c}_n(1 + k_n)d(w_n, p) + \bar{c}_n d(v_n, p)$   
and  $\{u_n\}$ ,  $\{v_n\}$ ,  $\{w_n\}$  are three bounded sequences in  $E$ .
- (b) there exists a constant  $M > 0$  such that  
 $d(x_m, p) \leq M d(x_n, p) + M \sum_{j=n}^{m-1} h_j$ ,  $\forall p \in F(T)$ ,  $m > n$ .

*Proof.* (a) Since  $T$  is asymptotically quasi-nonexpansive, for each  $p \in F(T)$ ,

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T^n y_n, u_n; a_n, b_n, c_n), p) \\ &\leq a_n d(x_n, p) + b_n d(T^n y_n, p) + c_n d(u_n, p) \\ &\leq a_n d(x_n, p) + b_n(1 + k_n) d(y_n, p) + c_n d(u_n, p), \end{aligned} \quad (2.1)$$

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T^n z_n, v_n; \bar{a}_n, \bar{b}_n, \bar{c}_n), p) \\ &\leq \bar{a}_n d(x_n, p) + \bar{b}_n d(T^n z_n, p) + \bar{c}_n d(v_n, p) \\ &\leq \bar{a}_n d(x_n, p) + \bar{b}_n(1 + k_n) d(z_n, p) + \bar{c}_n d(v_n, p) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T^n x_n, w_n; \hat{a}_n, \hat{b}_n, \hat{c}_n), p) \\ &\leq \hat{a}_n d(x_n, p) + \hat{b}_n d(T^n x_n, p) + \hat{c}_n d(w_n, p) \\ &\leq \hat{a}_n d(x_n, p) + \hat{b}_n(1 + k_n) d(x_n, p) + \hat{c}_n d(w_n, p). \end{aligned} \quad (2.3)$$

Substituting (2.3) into (2.2), we have

$$\begin{aligned} d(y_n, p) &\leq \bar{a}_n d(x_n, p) \\ &\quad + \bar{b}_n(1 + k_n) \{ \hat{a}_n d(x_n, p) + \hat{b}_n(1 + k_n) d(x_n, p) \\ &\quad + \hat{c}_n d(w_n, p) \} + \bar{c}_n d(v_n, p) \\ &= \bar{a}_n d(x_n, p) + \bar{b}_n \hat{a}_n (1 + k_n) d(x_n, p) \\ &\quad + \bar{b}_n \hat{b}_n (1 + k_n)^2 d(x_n, p) + \bar{b}_n \hat{c}_n (1 + k_n) d(w_n, p) \\ &\quad + \bar{c}_n d(v_n, p) \\ &\leq \bar{a}_n d(x_n, p) + \bar{b}_n \hat{a}_n (1 + k_n)^2 d(x_n, p) \\ &\quad + \bar{b}_n \hat{b}_n (1 + k_n)^2 d(x_n, p) + \theta_n \\ &\leq \bar{a}_n d(x_n, p) + \bar{b}_n (1 + k_n)^2 d(x_n, p) + \theta_n, \end{aligned} \quad (2.4)$$

where  $\theta_n = \bar{b}_n \hat{c}_n (1 + k_n) d(w_n, p) + \bar{c}_n d(v_n, p)$ . And again, substituting (2.4) into (2.1),

it follows that

$$\begin{aligned}
d(x_{n+1}, p) &\leq a_n d(x_n, p) \\
&\quad + b_n(1 + k_n) \{ \bar{a}_n d(x_n, p) + \bar{b}_n(1 + k_n)^2 d(x_n, p) + \theta_n \} \\
&\quad + c_n d(u_n, p) \\
&= a_n d(x_n, p) + b_n \bar{a}_n (1 + k_n) d(x_n, p) \\
&\quad + b_n \bar{b}_n (1 + k_n)^3 d(x_n, p) + b_n(1 + k_n) \theta_n + c_n d(u_n, p) \\
&= a_n d(x_n, p) + (1 - a_n - c_n) \bar{a}_n (1 + k_n) d(x_n, p) \\
&\quad + (1 - a_n - c_n) \bar{b}_n (1 + k_n)^3 d(x_n, p) + h_n \\
&\leq a_n d(x_n, p) + (1 - a_n) \bar{a}_n (1 + k_n)^3 d(x_n, p) \\
&\quad + (1 - a_n) \bar{b}_n (1 + k_n)^3 d(x_n, p) + h_n \\
&\leq a_n (1 + k_n)^3 d(x_n, p) \\
&\quad + (1 - a_n) (1 + k_n)^3 (\bar{a}_n + \bar{b}_n) d(x_n, p) + h_n \\
&\leq a_n (1 + k_n)^3 d(x_n, p) + (1 - a_n) (1 + k_n)^3 d(x_n, p) + h_n \\
&= (1 + k_n)^3 d(x_n, p) + h_n,
\end{aligned}$$

where  $h_n = b_n(1 + k_n)\theta_n + c_n d(u_n, p)$ . This completes the proof of (a).

(b) If  $a \geq 0$ , then  $1 + a \leq e^a$  and  $(1 + a)^3 \leq e^{3a}$ . Therefore, from (a) we can obtain that

$$\begin{aligned}
d(x_m, p) &\leq (1 + k_{m-1})^3 d(x_{m-1}, p) + h_{m-1} \\
&\leq e^{3k_{m-1}} d(x_{m-1}, p) + h_{m-1} \\
&\leq e^{3k_{m-1}} [(1 + k_{m-2})^3 d(x_{m-2}, p) + h_{m-2}] + h_{m-1} \\
&\leq e^{3(k_{m-1} + k_{m-2})} d(x_{m-2}, p) + e^{3k_{m-1}} h_{m-2} + h_{m-1} \\
&\leq e^{3(k_{m-1} + k_{m-2})} d(x_{m-2}, p) + e^{3k_{m-1}} (h_{m-1} + h_{m-2}) \\
&\leq \dots \\
&\leq e^{3 \sum_{j=n}^{m-1} k_j} d(x_n, p) + e^{3 \sum_{j=n}^{m-1} k_j} \sum_{j=n}^{m-1} h_j \\
&\leq M d(x_n, p) + M \sum_{j=n}^{m-1} h_j,
\end{aligned}$$

where  $M = e^{3 \sum_{j=n}^{\infty} k_j}$ . This completes the proof of (b).  $\square$

We also need the following lemma for the proof of our main results.

**Lemma 2.2** [8]. *Let the number of sequences  $\{a_n\}$ ,  $\{b_n\}$  and  $\{\lambda_n\}$  satisfy that  $a_n \geq 0$ ,  $b_n \geq 0$ ,  $\lambda_n \geq 0$ ,  $a_{n+1} \leq (1 + \lambda_n)a_n + b_n$ ,  $\forall n \in \mathbb{N}$ ,  $\sum_{n=1}^{\infty} b_n < \infty$ ,  $\sum_{n=1}^{\infty} \lambda_n < \infty$ . Then*

- (a)  $\lim_{n \rightarrow \infty} a_n$  exist.
- (b) if  $\liminf_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

Now, we are in a position to prove the main theorems.  $D_d(y, S)$  denotes the distance from  $y$  to set  $S$ , that is,  $D_d(y, S) = \inf\{d(y, s) : s \in S\}$ .

**Theorem 2.1.** *Let  $(E, d, W)$  be a complete convex metric space,  $T : E \rightarrow E$  be an asymptotically quasi-nonexpansive mapping and  $F(T)$  be a nonempty set. For a given  $x_1 \in E$ , let  $\{x_n\}$  be the three-step iterative sequence with errors defined by (1.1) and  $\{k_n\}, \{c_n\}, \{\bar{c}_n\}, \{\hat{c}_n\}$  be four sequences satisfying the following conditions :*

- (i)  $\sum_{n=1}^{\infty} k_n < \infty$ ,
- (ii)  $\sum_{n=1}^{\infty} c_n < \infty$ ,  $\sum_{n=1}^{\infty} \bar{c}_n < \infty$ ,  $\sum_{n=1}^{\infty} \hat{c}_n < \infty$ ,

where  $\{k_n\}$  is a sequence appeared in Definition 1.1 and  $\{c_n\}, \{\bar{c}_n\}, \{\hat{c}_n\}$  are three sequences appeared in (1.1). Then the iterative sequence  $\{x_n\}$  converges to a fixed point of  $T$  if and only if

$$\liminf_{n \rightarrow \infty} D_d(x_n, F(T)) = 0.$$

*Proof.* The necessity is obvious. Now, we prove the sufficiency. Suppose that the condition  $\liminf_{n \rightarrow \infty} D_d(x_n, F(T)) = 0$  is satisfied. Then from Lemma 2.1 (a), we have

$$d(x_{n+1}, p) \leq (1 + k_n)^3 d(x_n, p) + h_n, \quad \forall p \in F(T), \quad \forall n \in \mathbb{N}, \quad (2.5)$$

where  $h_n = b_n(1 + k_n)\theta_n + c_n d(u_n, p)$  and  $\theta_n = \bar{b}_n \hat{c}_n(1 + k_n)d(w_n, p) + \bar{c}_n d(v_n, p)$ . Since  $0 \leq b_n, \bar{b}_n \leq 1$ ,  $\sum_{n=1}^{\infty} k_n < \infty$ ,  $\sum_{n=1}^{\infty} \hat{c}_n < \infty$ ,  $\sum_{n=1}^{\infty} \bar{c}_n < \infty$ ,  $\sum_{n=1}^{\infty} c_n < \infty$  and  $\{w_n\}, \{v_n\}, \{u_n\}$  are three bounded sequences, we have  $\sum_{n=1}^{\infty} \theta_n < \infty$  and so  $\sum_{n=1}^{\infty} h_n < \infty$ . From (2.5), we can obtain that

$$D_d(x_{n+1}, F(T)) \leq (1 + k_n)^3 D_d(x_n, F(T)) + h_n.$$

Since  $\liminf_{n \rightarrow \infty} D_d(x_n, F(T)) = 0$ , by Lemma 2.2, we have

$$\lim_{n \rightarrow \infty} D_d(x_n, F(T)) = 0.$$

Now, we will prove that  $\{x_n\}$  is a Cauchy sequence. Let  $\epsilon > 0$ . By Lemma 2.1 (b), there exists a constant  $M > 0$  such that

$$d(x_m, p) \leq M d(x_n, p) + M \sum_{j=n}^{m-1} h_j, \quad \forall p \in F(T), \quad m > n. \quad (2.6)$$

Since  $\lim_{n \rightarrow \infty} D_d(x_n, F(T)) = 0$  and  $\sum_{n=1}^{\infty} h_n < \infty$ , there exists a constant  $N_1$  such that for all  $n \geq N_1$ ,

$$D_d(x_n, F(T)) < \frac{\epsilon}{4M} \quad \text{and} \quad \sum_{j=N_1}^{\infty} h_j < \frac{\epsilon}{6M}.$$

We note that there exists  $p_1 \in F(T)$  such that  $d(x_{N_1}, p_1) < \frac{\epsilon}{3M}$ . It follows from (2.6) that for all  $m > n \geq N_1$ ,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, p_1) + d(x_n, p_1) \\ &\leq Md(x_{N_1}, p_1) + M \sum_{j=N_1}^{m-1} h_j + Md(x_{N_1}, p_1) + M \sum_{j=N_1}^{n-1} h_j \\ &< M \frac{\epsilon}{3M} + M \frac{\epsilon}{6M} + M \frac{\epsilon}{3M} + M \frac{\epsilon}{6M} \\ &= \epsilon. \end{aligned} \quad (2.7)$$

Since  $\epsilon$  is an arbitrary positive number, (2.7) implies that  $\{x_n\}$  is a Cauchy sequence. From the completeness of this space,  $\lim_{n \rightarrow \infty} x_n$  exists. Let  $\lim_{n \rightarrow \infty} x_n = p$ . It will be proven that  $p$  is a fixed point. Let  $\bar{\epsilon} > 0$ . Since  $\lim_{n \rightarrow \infty} x_n = p$ , there exists a natural number  $N_2$  such that for all  $n \geq N_2$ ,

$$d(x_n, p) < \frac{\bar{\epsilon}}{2(2+k_1)}. \quad (2.8)$$

$\lim_{n \rightarrow \infty} D_d(x_n, F(T)) = 0$  implies that there exists a natural number  $N_3 \geq N_2$  such that for all  $n \geq N_3$ ,

$$D_d(x_n, F(T)) < \frac{\bar{\epsilon}}{3(4+3k_1)}.$$

Therefore, there exists a  $\bar{p} \in F(T)$  such that

$$d(x_{N_3}, \bar{p}) < \frac{\bar{\epsilon}}{2(4+3k_1)}. \quad (2.9)$$

From (2.8) and (2.9), we have

$$\begin{aligned} d(Tp, p) &\leq d(Tp, \bar{p}) + d(\bar{p}, Tx_{N_3}) + d(Tx_{N_3}, \bar{p}) + d(\bar{p}, x_{N_3}) + d(x_{N_3}, p) \\ &= d(Tp, \bar{p}) + 2d(Tx_{N_3}, \bar{p}) + d(x_{N_3}, \bar{p}) + d(x_{N_3}, p) \\ &\leq (1+k_1)d(p, \bar{p}) + 2(1+k_1)d(x_{N_3}, \bar{p}) + d(x_{N_3}, \bar{p}) \\ &\quad + d(x_{N_3}, p) \\ &\leq (1+k_1)\{d(p, x_{N_3}) + d(x_{N_3}, \bar{p})\} + 2(1+k_1)d(x_{N_3}, \bar{p}) \\ &\quad + d(x_{N_3}, \bar{p}) + d(x_{N_3}, p) \\ &= (2+k_1)d(x_{N_3}, p) + (4+3k_1)d(x_{N_3}, \bar{p}) \\ &< (2+k_1) \frac{\bar{\epsilon}}{2(2+k_1)} + (4+3k_1) \frac{\bar{\epsilon}}{2(4+3k_1)} \\ &= \bar{\epsilon}. \end{aligned}$$

Since  $\bar{\epsilon}$  is an arbitrary positive number, this implies that  $Tp = p$ , that is,  $p$  is a fixed point. This completes the proof of Theorem 2.1.  $\square$

In (1.1), if  $\hat{b}_n \equiv 0$  and  $\hat{c}_n \equiv 0$ , for all  $n = 1, 2, \dots$ , then  $z_n = x_n$ . Therefore, the following corollary can be obtained from Theorem 2.1 immediately.



**Corollary 2.1.** Let  $(E, d, W), T$  and  $F(T)$  be as in Theorem 2.1. For a given  $x_1 \in E$ , let  $\{x_n\}$  be the Ishikawa type iterative sequence with errors defined by (1.2) and  $\{k_n\}, \{c_n\}, \{\bar{c}_n\}$  be three sequences satisfying the conditions (i) and (ii) in Theorem 2.1. Then the iterative sequence  $\{x_n\}$  converges to a fixed point of  $T$  if and only if

$$\liminf_{n \rightarrow \infty} D_d(x_n, F(T)) = 0.$$

By using the same method in Theorem 2.1, we can easily obtain the following theorem.

**Theorem 2.2.** Let  $(E, d, W)$  be a complete convex metric space,  $T : E \rightarrow E$  be a quasi-nonexpansive mapping and  $F(T)$  be a nonempty set. For a given  $x_1 \in E$ , let  $\{x_n\}$  be the three-step iterative sequence with errors defined by (1.1) and  $\{k_n\}, \{c_n\}, \{\bar{c}_n\}, \{\hat{c}_n\}$  be four sequences satisfying the conditions (i) and (ii) in Theorem 2.1. Then the iterative sequence  $\{x_n\}$  converges to a fixed point of  $T$  if and only if

$$\liminf_{n \rightarrow \infty} D_d(x_n, F(T)) = 0.$$

**Theorem 2.3.** Let  $(E, d, W)$  be a complete convex metric space,  $T : E \rightarrow E$  be an asymptotically nonexpansive mapping and  $F(T)$  be a nonempty set. For a given  $x_1 \in E$ , let  $\{x_n\}$  be the three-step iterative sequence with errors defined by (1.1) and  $\{k_n\}, \{c_n\}, \{\bar{c}_n\}, \{\hat{c}_n\}$  be four sequences satisfying the conditions (i) and (ii) in Theorem 2.1. Then the iterative sequence  $\{x_n\}$  converges to a fixed point of  $T$  if and only if

$$\liminf_{n \rightarrow \infty} D_d(x_n, F(T)) = 0.$$

*Proof.* Since  $F(T)$  is a nonempty set, by Definition 1.1, an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive mapping. Thus, the conclusion can be obtained from Theorem 2.1 immediately.  $\square$

From Theorem 2.1, we can also obtain the following result for the Banach space.

**Theorem 2.4.** Let  $E$  be a real Banach space,  $T : E \rightarrow E$  be an asymptotically quasi-nonexpansive mapping satisfying the condition (i) in Theorem 2.1 and  $F(T)$  be a nonempty set. Let  $\{x_n\}$  be the three-step iterative sequence with errors defined by

$$\begin{cases} x_1 \in E, \\ x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \\ y_n = \bar{a}_n x_n + \bar{b}_n T^n z_n + \bar{c}_n v_n, \\ z_n = \hat{a}_n x_n + \hat{b}_n T^n x_n + \hat{c}_n w_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where  $\{u_n\}$ ,  $\{v_n\}$ ,  $\{w_n\}$  are three bounded sequences in  $E$  and  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{\bar{a}_n\}$ ,  $\{\bar{b}_n\}$ ,  $\{\bar{c}_n\}$ ,  $\{\hat{a}_n\}$ ,  $\{\hat{b}_n\}$ ,  $\{\hat{c}_n\}$  are nine sequences in  $[0, 1]$  satisfying  $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = \hat{a}_n + \hat{b}_n + \hat{c}_n = 1$ ,  $\forall n \in \mathbb{N}$  and  $\sum_{n=1}^{\infty} c_n < \infty$ ,  $\sum_{n=1}^{\infty} \bar{c}_n < \infty$ ,  $\sum_{n=1}^{\infty} \hat{c}_n < \infty$ . Then the iterative sequence  $\{x_n\}$  converges to a fixed point of  $T$  if and only if

$$\liminf_{n \rightarrow \infty} D(x_n, F(T)) = 0,$$

where  $D(y, S) = \inf\{\|y - s\| : s \in S\}$ .

*proof.* Since  $E$  is a Banach space, it is a complete convex metric space with a convex structure  $W(x, y, z : \alpha, \beta, \gamma) := \alpha x + \beta y + \gamma z$ , for all  $x, y, z \in E$  and for all  $\alpha, \beta, \gamma \in [0, 1]$  with  $\alpha + \beta + \gamma = 1$ . Therefore, the conclusion of Theorem 2.4 can be obtained from Theorem 2.1 immediately.  $\square$

**Remark 2.1.** (1) Theorem 2.1, 2.2 and 2.3 are three new convergence theorems of three-step iterative sequences with errors for nonlinear mappings in convex metric spaces. These three theorems generalize and improve the corresponding results of [7-9] and [1-3, 5, 11, 14].

(2) Theorem 2.4 generalizes and improves the corresponding results of Kim *et al.* [6], Liu [8, 9] and Xu-Noor [14].

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